

Phase transition in matrix model with logarithmic action: toy-model for gluons in baryons

Govind S. Krishnaswami

Institute for Theoretical Physics & Spinoza Institute

Utrecht University, Postbus 80.195, 3508 TD, Utrecht, The Netherlands

E-mail: g.s.krishnaswami@phys.uu.nl

ABSTRACT: We study the competing effects of gluon self-coupling and their interactions with quarks in a baryon, using the very simple setting of a hermitian 1-matrix model with action $\text{tr}A^4 - \log \det(\nu + A^2)$. The logarithmic term comes from integrating out N quarks. The model is a caricature of 2d QCD coupled to adjoint scalars, which are the transversely polarized gluons in a dimensional reduction. ν is a dimensionless ratio of quark mass to coupling constant. The model interpolates between gluons in the vacuum ($\nu = \infty$), gluons weakly coupled to heavy quarks (large ν) and strongly coupled to light quarks in a baryon ($\nu \rightarrow 0$). Its solution in the large- N limit exhibits a phase transition from a weakly coupled 1-cut phase to a strongly coupled 2-cut phase as ν is decreased below $\nu_c = 0.27$. Free energy and correlation functions are discontinuous in their third and second derivatives at ν_c . The transition to a two-cut phase forces eigenvalues of A away from zero, making glue-ring correlations grow as ν is decreased. In particular, they are enhanced in a baryon compared to the vacuum. This investigation is motivated by a desire to understand why half the proton's momentum is contributed by gluons.

KEYWORDS: 1/N Expansion, Matrix Models, Field Theories in Lower Dimensions, QCD.

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1. Introduction

Photons carry a negligible amount of momentum in a moving atom. By contrast, it is experimentally found [1] that gluons carry about half the momentum of a nucleon when observed at momentum transfers of order $Q^2 \sim 1 \text{ (GeV)}^2$. The growth of the gluon momentum contribution as Q^2 is increased is correctly predicted by perturbative QCD. However, the momentum fraction (x_{Bj}) dependence of gluons in a nucleon at any fixed value of Q^2 appears to be essentially non-perturbative. More generally, determining the ‘emergent’ bound-state structure of gluons in baryons from QCD remains an interesting and challenging open problem of theoretical physics. This problem in 3 or 4 dimensional QCD is still very hard to address analytically. However, there is a context where it can at least be finitely formulated. This is in the class of theories where 2d QCD is coupled to adjoint scalar fields. The principal examples are the reduction of QCD from 3 or 4 to 2 dimensions,

where we assume that fields are independent of transverse coordinates.¹ In this case, the longitudinal component of the gauge field contributes a linear potential between dynamical quarks $q^a(x)$ and the transverse components transform as adjoint scalar fields $A_b^a(x)$, for instance in the light-cone gauge. Thus, for N colors a, b , we have a two-dimensional field theory of $N \times N$ hermitian matrix-valued and N -component vector-valued fields with only a ‘global’ $U(N)$ invariance. The gauge-invariant observables include glue-ring (closed-string) variables

$$\text{tr}[A(x_1)A(x_2) \cdots A(x_n)] \tag{1.1}$$

and meson (open-string) variables

$$q_{a_1}^\dagger(x_0) A_{a_2}^{a_1}(x_1) A_{a_3}^{a_2}(x_2) \cdots A_{a_{n+1}}^{a_n}(x_n) q^{a_{n+1}}(x_{n+1}) \tag{1.2}$$

where the points x_i are null separated.² In ’t Hooft’s large- N limit, the expectation values of products of these variables factorize. So we may restrict attention to the single trace observables. Quarks are suppressed by one power of N in calculating vacuum expectation values of glue-ring observables in the large- N limit [2]. However, when the vacuum is replaced by a baryon state, interactions with quarks are just as important as gauge field self-interactions [3]. For definiteness, let us consider 2d QCD coupled to a single adjoint scalar field via the action (non-dynamical fields have been eliminated)

$$S = \int dt dx \text{tr} \left[q^\dagger i \partial_t q + \frac{1}{2} m^2 q^\dagger \frac{1}{i \partial_x} q + \partial_x A \partial_t A \right] - \frac{g^2}{2} \int dt dx dy J_b^a(x) \frac{|x-y|}{2} J_a^b(y) \tag{1.3}$$

where the current

$$J_b^a(x) = i[A(x), \partial_x A(x)]_b^a - q_b^\dagger(x) q^a(x). \tag{1.4}$$

m is the ‘current’ quark mass and g is a coupling constant with the dimensions of mass. There is no term of the form $q^\dagger A q$ when non-dynamical fields are eliminated. This is essentially the dimensional reduction from 3 dimensions, except for the absence of terms of the form $q^\dagger A \frac{1}{\partial_x} A q$ and $q^\dagger A \frac{1}{\partial_x} q$. These theories have no ultraviolet divergences, though m undergoes a finite renormalization ($m^2 \rightarrow m^2 - g^2 N/\pi$). Heuristically, g^2 is related to the dimensionless 4d coupling constant g_4^2 via a factor of the transverse area of hadrons, $g^2 \sim \Lambda_{\text{QCD}}^2 g_4^2$.

Before proceeding further, let us mention some literature on these theories. The adjoint scalar glueball spectrum was studied by numerical diagonalization of the hamiltonian by Klebanov and Dalley [4]. There is a large literature on discretized light-cone and transverse lattice approaches for which we refer to the review by Burkardt and Dalley [5]. Among the more analytical approaches, Rajeev, Turgut and Lee [6] studied the Poisson and Lie algebras of Wilson loops, open and closed string observables in the large- N limit of the

¹This dimensional reduction may well provide a first approximation to the gluon distribution of the proton measured in the Bjorken limit of deep inelastic scattering.

²The parallel transport operators between these points are trivial in a gauge where the null component of the the gauge field is set to zero.

hamiltonian framework. On the other hand, Rajeev formulated 2 dimensional QCD in the large- N limit (without the adjoint scalars) as a non-linear classical theory of gauge-invariant quark bilinears [7]. By developing approximation methods to solve this non-linear classical theory it was possible to analytically determine the quark structure of the baryon predicted by 2d QCD. This was shown to provide a first approximation to the non-perturbative quark structure functions of baryons measured in deep inelastic scattering [8, 9].

Despite these developments, determining the expectation values of glue-ring and meson observables analytically remains quite hard for two reasons. On the one hand, there are an infinite number of observables with ever increasing string length. Their dynamics is inter-linked via the factorized Schwinger-Dyson or loop equations. On the other hand, we are interested not just in the vacuum correlations but those in the ground state of the baryon, which has itself to be determined dynamically.

We propose to use the approximate ground-state of the baryon $|\Psi\rangle$ determined in the large- N limit of 2d QCD as a starting point for the harder problem of determining the gluon correlations in the baryon. We have shown [8, 9] that

$$|\Psi\rangle = \int dx_1 \cdots dx_N \epsilon^{a_1 \cdots a_N} \psi(x_1, \dots, x_N) q_{a_1}^* \cdots q_{a_N}^*(x_N) \quad (1.5)$$

for a factorized wave function (see section 2) provides a good approximation to the ground state of the baryon in the chiral and large- N limits. In a sense, the state $|\Psi\rangle$ contains no gluons. Nevertheless, the presence of a baryon $|\Psi\rangle$ made of N quarks may be expected to deform the vacuum of the gluon field. The idea is to approximately determine the large- N expectation values of glue-ring observables (1.1) in the background of this N quark baryon. Though $|\Psi\rangle$ is not the true baryon ground state of the action (1.3), it is still an approximate ground state of the baryon number one sector of 2d QCD in the large- N limit. As such, it furnishes a variational approximation to the ground state of (1.3). In this manner, we hope to isolate a portion of this difficult problem that we have some chance of addressing analytically.

In this paper we will take a very modest step towards the above-mentioned goal by studying a much simplified version via the path integral approach in the large- N limit. To focus on the interesting baryon-part of the problem, we study a caricature of the action (1.3) without derivatives or non-local interactions and where fields are assumed time independent:

$$S(A, q, q^*) = \text{tr} \int dx [\alpha A^4(x) - iq^*(x)(m + \alpha A^2(x))q(x)]. \quad (1.6)$$

Here α is the 't Hooft coupling with dimensions of mass, held fixed as $N \rightarrow \infty$. Though an oversimplification, it allows us to isolate the dynamics associated to the degrees of freedom at the locations of the N quarks making up the baryon. After integrating out the quarks we get an N matrix model with an action involving not just traces of powers of the adjoint scalar, but also its inverse and determinant. To get a computable toy-model that preserves the essential picture of gluon correlations in a baryon background, we assume that the adjoint scalar field is equal at the positions of the different quarks. This leads us to a

1-matrix model whose action is a quartic polynomial with a logarithmic modification.

$$S(A) = \text{tr} [A^4 - \log [\nu + A^2]] \tag{1.7}$$

where the dimensionless parameter ν is a ratio of quark mass to coupling constant. The logarithmic term encapsulates the effect of the N -quark baryon. Due to the significant truncations we make, this matrix model is not an approximation to the 1 + 1 dimensional field theory. It is the simplest toy-model where we may study the competing effects of gluon self-interactions and their interaction with quarks in a baryon. Finite-dimensional matrix models are often of independent interest. For instance, a slightly different logarithmic action $\log(1 - A) + A$ appears in the Penner matrix model, which found applications to $c = 1$ string theory [10].

Summary of Results: We solve our 1-matrix model in the large- N limit by an extension of the methods used by Brezin et. al. [11] for polynomial potentials. This is possible because the derivative of the action is a rational function. We determine the free energy and glue-ring expectation values (moments of the adjoint scalar) in this caricature of a baryon. As ν decreases below $\nu_c = 0.27$, the 1-cut solution³ of the matrix model makes a transition to a 2-cut solution. The reason is that the logarithmic term in the action (due to the presence of the baryon) makes small eigenvalues for the adjoint scalar energetically costly. The 1-cut solution could have been obtained by summing planar diagrams, but the 2-cut solution is analytically unrelated and could not be obtained that way. At the critical point ν_c , the leading discontinuity in free energy is in its 3rd derivative, while the 2nd derivatives of the two- and four-point correlations are discontinuous. This large- N phase transition bears some resemblance to the Gross-Witten transition in the unitary one-plaquette model [12] or that in the hermitian $m^2 A^2 + g A^4$ matrix model as m^2/\sqrt{g} is made sufficiently negative [13]. All these phase transitions involve a jump discontinuity in the third derivative of free energy. One consequence is that analytic continuation from a weak-coupling treatment of gluons, starting from the vacuum ($\nu = \infty$) would not be successful in predicting their behavior when strongly coupled to light quarks in a baryon ($\nu \rightarrow 0^+$). Moreover, moments of the adjoint scalar are enhanced in the baryon compared to the vacuum, since, in the two-cut phase, the support of the eigenvalue distribution excludes the origin.

Organization of article. We begin with our ansatz for the baryon state in section 2 and give a path integral formulation of the problem of finding glue-ring correlations in section 3. Section 4 describes the truncations and approximations that lead us to a matrix model for gluon correlations in a baryon. Sections 5 and 6 give the solution of the matrix model in the weak-coupling ($\nu \geq \nu_c$) and strong-coupling ($\nu \leq \nu_c$) phases respectively. The heavy quark limit, neighborhood of the critical point, and the chiral limit are treated in section 7. Section 8 presents a discussion of our results and open questions.

³One and two ‘cuts’ refer to the number of disjoint intervals of the real line on which the distribution of eigenvalues of A is supported.

2. Ansatz for baryon state

A lesson from our study of 2d QCD in the large- N limit is that the ground state of the baryon is well approximated by a state containing N ‘valence’ quarks. Such a state may be built out of

$$| \hat{a}_1^{x_1}; \dots; \hat{a}_N^{x_N} \rangle = \hat{q}_{a_1}^\dagger(x_1) \hat{q}_{a_2}^\dagger(x_2) \dots \hat{q}_{a_N}^\dagger(x_N) |0\rangle \quad (2.1)$$

where \hat{q}, \hat{q}^\dagger satisfy canonical anti-commutation relations with respect to the Dirac vacuum $|0\rangle$. Ignoring flavor and spin quantum numbers, a general N -quark state is a linear combination

$$|\Psi\rangle = \hat{\Psi}|0\rangle = \int dx_1 \dots dx_N \Psi(\hat{a}_1^{x_1}; \dots; \hat{a}_N^{x_N}) \hat{q}_{a_1}^\dagger(x_1) \hat{q}_{a_2}^\dagger(x_2) \dots \hat{q}_{a_N}^\dagger(x_N) |0\rangle. \quad (2.2)$$

To represent a baryon, Ψ must be totally anti-symmetric in color

$$\Psi(\hat{a}_1^{x_1}; \dots; \hat{a}_N^{x_N}) = \epsilon^{a_1 \dots a_N} \psi(x_1, \dots, x_N), \quad (2.3)$$

and therefore represents a fermion if $\psi(x_1, \dots, x_N)$ is a symmetric function. For a path integral formulation, associate the grassmann-valued field $q^a(x)$ and its complex conjugate $q_a^*(x)$ to the operators $\hat{q}^a(x)$ and $\hat{q}_a^\dagger(x)$. Then the baryon state $|\Psi\rangle$ becomes

$$|\Psi\rangle \mapsto \int dx_1 \dots dx_N \epsilon^{a_1 \dots a_N} \psi(x_1 \dots x_N) q_{a_1}^*(x_1) q_{a_2}^*(x_2) \dots q_{a_N}^*(x_N). \quad (2.4)$$

By solving 2d QCD in the large- N limit we determined that $\psi(x_1 \dots x_N)$ is well-approximated by an N -fold product of single particle wave-functions $\psi(x)$. A way to understand this is that once the antisymmetry in color is accounted for, quarks behave like N bosons which condense to the same one-particle ground-state. A good variational estimate turns out to be $\psi(x) = \frac{1}{\sqrt{\pi}} \frac{1}{(1-ix)}$ which is the Fourier transform of the ‘valence quark wave-function’ $\tilde{\psi}(p) = 2\sqrt{\pi} e^{-p} \theta(p \geq 0)$. Though presented in the language of quantum many-body theory, the above ansatz may be obtained as the first term in a systematic field theoretic approximation method for the ground state of 2d QCD (see [9] for details). One goal is to have a similar understanding of the gluon content of the baryon from the adjoint scalar field theory (1.3), which may then also be interpreted in terms of a relativistic many-body problem.

3. Gluon correlations in the baryon state

For any $U(N)$ -invariant operator \mathcal{O} , such as the glue-ring (1.1), the relation between expectation values of Heisenberg field operators and functional integrals in Euclidean space-time is

$$\begin{aligned} \langle \xi | \mathcal{O}(A, q, q^*) | \xi \rangle &= \langle 0 | \hat{\xi} \mathcal{O} \hat{\xi}^\dagger | 0 \rangle = \frac{1}{Z} \int \mathcal{D}A \mathcal{D}q \mathcal{D}q^* e^{-NS(A, q, q^*)} \xi \mathcal{O}(A, q, q^*) \xi^* \\ \text{where } Z &= \int \mathcal{D}A \mathcal{D}q \mathcal{D}q^* e^{-S(A, q, q^*)} \xi \xi^* \end{aligned} \quad (3.1)$$

Here ξ^\dagger creates a baryon in the infinite past $-T \rightarrow -\infty$ while ξ annihilates it in the infinite future T . Using periodic boundary conditions in time we can assume that these operators are adjoints of each other and evaluated at a common time T . Specializing to our baryon state $|\xi\rangle = |\Psi\rangle$ given in (2.4),

$$\begin{aligned} \langle \Psi | \mathcal{O}(A, q, q^*) | \Psi \rangle &= \frac{1}{Z} \int \mathcal{D}A \mathcal{D}q \mathcal{D}q^* e^{-NS(A, q, q^*)} \int dx_1 \cdots dx_N \epsilon_{a_1 \cdots a_N} \psi(x_1, \dots, x_N) \\ &\quad \times q^{a_1}(x_1, T) \cdots q^{a_N}(x_N, T) \mathcal{O}(A, q, q^*) \int dy_1 \cdots dy_N \epsilon^{b_1 \cdots b_N} \\ &\quad \times \psi^*(y_1, \dots, y_N) q_{b_1}^*(y_1, T) \cdots q_{b_N}^*(y_N, T) \end{aligned} \quad (3.2)$$

The functional integrals over Grassmann variables would vanish if $x_i \neq y_i$, so we take them to be equal and get

$$\begin{aligned} \langle \Psi | \mathcal{O}(A, q, q^*) | \Psi \rangle &= \frac{1}{Z} \int dx_1 \cdots dx_N |\psi(x_1, \dots, x_N)|^2 \int \mathcal{D}A \mathcal{D}q \mathcal{D}q^* e^{-NS(A, q, q^*)} \epsilon_{a_1 \cdots a_N} \epsilon^{b_1 \cdots b_N} \\ &\quad \times q^{a_1}(x_1, T) \cdots q^{a_N}(x_N, T) \mathcal{O}(A, q, q^*) q_{b_1}^*(x_1, T) \cdots q_{b_N}^*(x_N, T) \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} Z &= \int dx_1 \cdots dx_N |\psi(x_1, \dots, x_N)|^2 \int \mathcal{D}A \mathcal{D}q \mathcal{D}q^* e^{-NS(A, q, q^*)} \epsilon_{a_1 \cdots a_N} \epsilon^{b_1 \cdots b_N} \\ &\quad \times q^{a_1}(x_1, T) \cdots q^{a_N}(x_N, T) q_{b_1}^*(x_1, T) \cdots q_{b_N}^*(x_N, T) \\ &= \int dx_1 \cdots dx_N |\psi(x_1, \dots, x_N)|^2 Z(x_1, \dots, x_N) \end{aligned} \quad (3.4)$$

Now, we set aside the integral over x_1, \dots, x_N and focus on the functional integral⁴

$$\begin{aligned} Z(x_1, \dots, x_N) &= \int \mathcal{D}[Aqq^*] e^{-NS(A, q, q^*)} \epsilon_{a_1 \cdots a_N} \epsilon^{b_1 \cdots b_N} \\ &\quad q^{a_1}(x_1, T) \cdots q^{a_N}(x_N, T) q_{b_1}^*(x_1, T) \cdots q_{b_N}^*(x_N, T) \end{aligned} \quad (3.5)$$

4. Matrix model caricature of baryon

Physically, we are primarily interested in the shape and energy of the proton in its stationary ground state, which is probably best addressed in a hamiltonian framework. To mimic this in our path-integral approach, we will simply assume that fields are independent of time and ignore the ∂_t terms in (1.3), which contain the symplectic structure in a hamiltonian approach. This can be regarded either as a crude passage to a kind of hamiltonian or as a further dimensional reduction. Moreover, it is clear that the locations of the quarks x_1, \dots, x_N play a special role in the path integral (3.5). We would like to isolate the contribution of those points. Of course, this is not possible with the Lagrangian (1.3), since it involves spatial derivatives as well as non-local interactions. However, in keeping with the modest goals of this paper, we work with an ‘action’ that is motivated by (1.3), but does not involve any derivatives or non-local interactions

$$S(A, q, q^*) = \text{tr} \int dx \left\{ V(A) - iq^*(m + \alpha A^2)q \right\} \quad (4.1)$$

⁴One complication that we do not address in this paper is that in calculating expectation values, the functional integral is not separately normalized, but only after integration over the points x_1, \dots, x_N .

In this caricature, $V(A) = \alpha A^4$ stands in place of the current-current self-interaction of the adjoint scalars in (1.3). $q^* A^2 q$ models the quark-gluon cross term in the current-current interaction. We ignore the quartic term in quarks, which does not involve gluons. The primary effect of this term has already been taken into account in determining the baryon state $|\Psi\rangle$ (2.4) by solving 2d QCD in the large- N limit [7]. Thus (3.5) becomes

$$Z(x_1, \dots, x_N) = \int \mathcal{D}A \mathcal{D}q^* \mathcal{D}q e^{-N \int dx [\text{tr} V(A) - i q^* (m + \alpha A^2) q]} \times \epsilon^{a_1 \dots a_N} \epsilon_{b_1 \dots b_N} q^{b_N}(x_N) \dots q^{b_1}(x_1) q_{a_1}^*(x_1) \dots q_{a_N}^*(x_N) \quad (4.2)$$

The partition function can be written as:

$$Z(x_1, \dots, x_N) = \int \mathcal{D}A e^{-N \text{tr} \int dx V(A)} Z_q \quad (4.3)$$

where Z_q is the ‘quark part’ of the partition function

$$\begin{aligned} Z_q &= \int \mathcal{D}q^* \mathcal{D}q e^{iN \int dx q^* (m + \alpha A^2) q} \epsilon^{a_1 \dots a_N} \epsilon_{b_1 \dots b_N} q^{b_N}(x_N) \dots q^{b_1}(x_1) q_{a_1}^*(x_1) \dots q_{a_N}^*(x_N) \\ &= \prod_x \int dq^*(x) dq(x) e^{iN \Delta q^* (m + \alpha A^2) q} \epsilon^{a_1 \dots a_N} \epsilon_{b_1 \dots b_N} q^{b_N}(x_N) \dots q^{b_1}(x_1) q_{a_1}^*(x_1) \dots q_{a_N}^*(x_N). \end{aligned} \quad (4.4)$$

In the second line we have discretized $\int dx$ to a sum $\sum_x \Delta$, where $\Delta \sim dx$. Moreover, Z_q may be factored into a ‘vacuum part’ and a ‘baryon part’ $Z_q = Z_{qv} Z_{qb}$.

$$\begin{aligned} Z_{qv} &= \prod_{x \neq x_i} \int dq^*(x) dq(x) e^{iN \Delta q^* (m + \alpha A^2) q} = \prod_{x \neq x_i} \det [N \Delta (m + \alpha A^2(x))] \\ Z_{qb} &= \epsilon^{a_1 \dots a_N} \epsilon_{b_1 \dots b_N} \prod_{l=1}^N \int dq^*(x_l) dq(x_l) e^{iN \Delta q^* (m + \alpha A^2) q} q^{b_l}(x_l) q_{a_l}^*(x_l). \end{aligned} \quad (4.5)$$

Now we use the result $\int dq^* dq e^{i q^* B q} q^b q_a^* = [B^{-1}]_a^b \det [B]$ to get

$$\frac{Z_{qb}}{(N \Delta)^{N^2 - N}} = \epsilon^{a_1 \dots a_N} \epsilon_{b_1 \dots b_N} \det B(x_1) [B^{-1}(x_1)]_{a_1}^{b_1} \dots \det B(x_N) [B^{-1}(x_N)]_{a_N}^{b_N} \quad (4.6)$$

where $B(x_i) = m + \alpha A^2(x_i)$ is an $N \times N$ matrix for each $i = 1, \dots, N$. We can ignore the constant overall factor involving Δ . Thus Z_{qb} depends on the gluon field at the location of the quarks. We factor the integral over A into a baryon contribution and vacuum contribution.

$$\begin{aligned} \int \mathcal{D}A e^{-N \text{tr} \int dx V(A)} &= \prod_{x \neq x_i} \int dA(x) e^{-N \Delta \text{tr} V(A)} \\ &\times \int \prod_{i=1}^N dA(x_i) e^{-N \Delta \text{tr} \sum_{i=1}^N V(A(x_i))}. \end{aligned} \quad (4.7)$$

Combining the quark and gluon vacuum contributions, we define the vacuum part of the partition function as

$$Z_{\text{vac}}(x_1, \dots, x_N) = \prod_{x \neq x_i} \int dA(x) e^{-N \Delta \text{tr} V(A)} \det [N \Delta (m + \alpha A^2(x_i))]. \quad (4.8)$$

Z_{vac} is, for our purposes, the uninteresting part, so we consider the quotient

$$\frac{Z(x_1 \cdots x_N)}{Z_{\text{vac}}(x_1, \dots, x_N)} = \int \prod_{i=1}^N dA(x_i) e^{-N \text{tr} \sum_{i=1}^N \Delta V(A(x_i))} \epsilon^{a_1 \cdots a_N} \epsilon_{b_1 \cdots b_N} \times \det B(x_1) [B^{-1}(x_1)]_{a_1}^{b_1} \cdots \det B(x_N) [B^{-1}(x_N)]_{a_1}^{b_1} \quad (4.9)$$

Here $B(x_i) = m + \alpha A^2(x_i)$. This is an N -matrix model involving the $N \times N$ matrices $A(x_i)$, $i = 1, \dots, N$. We do not yet have a way of approximately solving this multi-matrix model, so we will make some further assumptions that allow us to reduce it to a one-matrix model.

4.1 Reduction to a 1-matrix model

Suppose we assume that the adjoint scalar gluon field is equal at the positions of the N quarks, $A(x_i) = A$ for $i = 1, \dots, N$. Then $B(x_1) = B(x_2) = \cdots = B(x_N) \equiv B$ and we can simplify Z_{qb} :

$$Z_{qb} = (\det B)^N \epsilon_{a_1 \cdots a_N} \epsilon^{b_1 \cdots b_N} [B^{-1}]_{b_1}^{a_1} \cdots [B^{-1}]_{b_N}^{a_N} = (\det B)^{N-1}. \quad (4.10)$$

Then,

$$\begin{aligned} \frac{Z(x_1 \cdots x_N)}{Z_{\text{vac}}(x_1, \dots, x_N)} &= \int dA e^{-NL \text{tr} V(A)} (\det B)^{N-1} \\ &= \int dA e^{-N[L \text{tr} V(A) - \log \det[m + \alpha A^2]]} \end{aligned} \quad (4.11)$$

where we have replaced $N - 1$ by N in anticipation of the large- N limit. $N\Delta \equiv L$ is a ‘length’ of the baryon, assumed to have a finite limit as $N \rightarrow \infty$ and $\Delta \rightarrow 0$. Recalling that $V(A) = \alpha A^4$, the action becomes $S(A) = \text{tr}[L\alpha A^4 - \log(m + \alpha A^2)]$. Re-scaling $A \rightarrow A/(\alpha L)^{1/4}$ we find there is only one dimensionless parameter $\nu = m\sqrt{L}/\sqrt{\alpha}$ on which the observables of this matrix model depend non-trivially. The appearance of L is an artifact of our truncation and could probably be avoided. In the field theory (1.3), the relevant parameter would be m/g . Thus we arrive at the one-matrix model

$$Z = \int dA e^{-N \text{tr}[A^4 - \log[\nu + A^2]]}. \quad (4.12)$$

ν as a dimensionless ratio of quark mass to coupling constant. The absence of a quadratic term in A may be traced to the absence of a gluon mass term in (1.3). Since A is hermitian, A^2 is positive. Thus, there is no difficulty in defining $\log[\nu + A^2]$ for $\nu > 0$. The expectation values of $\text{tr} A^n$ in this matrix model are what remain of the glue-ring expectation values.

5. One-cut solution of quartic + log matrix model

The rest of this paper will be devoted to a study of the large- N limit of the one-matrix model with action and free energy

$$S(A) = \text{tr} [A^4 - \log(\nu + A^2)], \quad E(\nu) = - \lim_{N \rightarrow \infty} \frac{1}{N^2} \log \int dA e^{-NS(A)}. \quad (5.1)$$

Glue-ring expectation values

$$G_n(\nu) = \lim_{N \rightarrow \infty} \left(\int dA e^{-NS(A)} \frac{1}{N} \text{tr} A^n \right) / \int dA e^{-NS(A)} \quad (5.2)$$

are given by moments $G_n = \int \rho(x) x^n dx$ of the eigenvalue density. $\rho(x)$ must minimize the free energy

$$E[\rho] = \int S(x) \rho(x) dx - \int dx dy \rho(x) \rho(y) \log |x - y|; \quad E(\nu) = \min_{\rho} E[\rho] \quad (5.3)$$

where $S(x) = x^4 - \log(\nu + x^2)$. (a) For ν sufficiently large, $S(x)$ is convex from below and we expect ρ to be supported on a single interval. (b) For ν sufficiently small (but positive), $S(x)$ is shaped like a Mexican hat (see figure 6) and we expect ρ to be supported on a pair of intervals located near the minima of $S(x)$. These two cases are treated in section 5 and 6. For ρ supported on a single interval, the Mehta-Dyson linear integral equation for an extremum of $E[\rho]$ is

$$S'(x) = 4x^3 - \frac{2x}{\nu + x^2} = 2\mathcal{P} \int_{-2a}^{2a} \frac{\rho(y)}{x - y} dy \quad -2a \leq x \leq 2a, \quad a > 0. \quad (5.4)$$

ρ is subject to positivity $\rho(x) \geq 0$ and normalization $\int_{-2a}^{2a} \rho(y) dy = 1$ conditions. Since $S(x)$ is even, ρ must be even. It is convenient to introduce the generating function of moments

$$F(z) = \int_{-2a}^{2a} \frac{\rho(y)}{z - y} dy = \sum_{n=0}^{\infty} \frac{G_n}{z^{n+1}}. \quad (5.5)$$

(a) $F(z)$ is analytic on $\mathbf{C} \setminus [-2a, 2a]$. (b) $F(z) \sim \frac{1}{z}$ as $|z| \rightarrow \infty$ which follows from normalization. (c) $F(z)$ is real for real z outside $[-2a, 2a]$. (d) When z approaches $[-2a, 2a]$, using (5.4),

$$F(x \pm i\epsilon) = \frac{1}{2} \left(4x^3 - \frac{2x}{\nu + x^2} \right) \mp i\pi \rho(x) \Rightarrow \rho(x) = \frac{1}{2\pi i} (F(x - i\epsilon) - F(x + i\epsilon)) \quad (5.6)$$

(e) $F(z)$ is an odd function of z since ρ is even. By analogy with the case of polynomial $S(x)$ we expect that if there is an $F(z)$ satisfying these conditions, it is unique. Existence, however, is not guaranteed. Indeed, we do not expect a 1-cut solution for sufficiently small $\nu > 0$. In a region of validity which is to be determined, the following ansatz for $F(z)$ is consistent with the above requirements

$$F(z) = 2z^3 - \frac{z}{\nu + z^2} + R(z) \sqrt{z^2 - 4a^2}. \quad (5.7)$$

$R(z)$ is a rational function to be chosen so that these conditions are satisfied. In particular, we must pick $R(z)$ to cancel the poles at $z = \pm i\sqrt{\nu}$ coming from $\frac{2z}{\nu + z^2}$. So $R(z) = \frac{P(z)}{(\nu + z^2)}$, where the polynomial $P(z)$ does not have zeros at $z = \pm i\sqrt{\nu}$. Moreover, to cancel the linear and cubic terms in $S'(z)$ as $|z| \rightarrow \infty$, we need to pick $P(z)$ to be a quartic even polynomial. Thus

$$F(z) = 2z^3 - \frac{z}{\nu + z^2} + \frac{\alpha + \gamma z^2 + \epsilon z^4}{(\nu + z^2)} \sqrt{z^2 - 4a^2}. \quad (5.8)$$

We need to determine the parameters $a, \alpha, \gamma, \epsilon$. Though it may appear that the third term is even in z , it is actually odd due to the square root. As $z \rightarrow \infty$,

$$F(z) \rightarrow (2 + \epsilon)z^3 + \left[\gamma - \epsilon(2a^2 + \nu) \right] z + \left[-1 + \alpha - 2a^4\epsilon - \gamma\nu + \epsilon\nu^2 - 2a^2(\gamma - \epsilon\nu) \right] \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^3}\right). \quad (5.9)$$

Requiring $F(z) \sim \frac{1}{z}$, fixes α, γ and ϵ

$$\epsilon = -2; \quad \gamma = -(2\nu + 4a^2); \quad \alpha = 2 - 12a^4 - 4\nu a^2. \quad (5.10)$$

To fix a we must require analyticity of $F(z)$ as $z^2 \rightarrow -\nu$. This is ensured if

$$-z + (\alpha + \gamma z^2 + \epsilon z^4) \sqrt{z^2 - 4a^2} \quad (5.11)$$

vanishes as $z \rightarrow \pm i\sqrt{\nu}$. In other words,

$$\pm i\sqrt{\nu} + (\alpha - \gamma\nu + \epsilon\nu^2) \sqrt{-\nu - 4a^2} = 0 \quad (5.12)$$

or $-\nu + (-2 + 12a^4)^2(4a^2 + \nu) = 0$, which is a quintic equation for $s = a^2$,

$$576s^5 + 144\nu s^4 - 192s^3 - 48\nu s^2 + 16s + 3\nu = 0. \quad (5.13)$$

We are guaranteed at least one real solution for a^2 . The physically allowed solutions are those with $a^2 > 0$. For $\nu > 0$, we find that there are two solutions $a^2 > 0$ only the smaller of which leads to $\rho(x)$ normalized to 1 (the other has ρ normalized to 3). For this value of a , the density of eigenvalues is

$$\rho(x) = -\left(\frac{1}{\pi}\right) R(x) \sqrt{4a^2 - x^2} = -\left(\frac{1}{\pi}\right) \frac{(\alpha + \gamma x^2 + \epsilon x^4) \sqrt{4a^2 - x^2}}{(\nu + x^2)} \quad (5.14)$$

with α, γ, ϵ given in (5.10).

5.1 Moments

The analogue of glue-ring expectation values (5.2) can be obtained from the Laurent series for the moment generating function $F(z)$

$$F(z) = 2z^3 - \frac{z}{\nu + z^2} + \frac{\alpha + \gamma z^2 + \epsilon z^4}{(\nu + z^2)} \sqrt{z^2 - 4a^2} = \sum_{n=0}^{\infty} \frac{G_n}{z^{n+1}} \quad (5.15)$$

The odd moments vanish $G_{2n+1} = 0$ and the even ones are

$$\begin{aligned} G_2 &= 40a^6 + 12\nu a^4 - 4a^2 - \nu, \\ G_4 &= 60a^8 - 24\nu a^6 - 4(1 + 3\nu^2)a^4 + 4\nu a^2 + \nu^2, \quad \text{etc.} \end{aligned} \quad (5.16)$$

where $a(\nu) = \sqrt{s}$ is the physical solution of (5.13). $G_2(\nu)$ and $G_4(\nu)$ are plotted in figure 1 and 3.

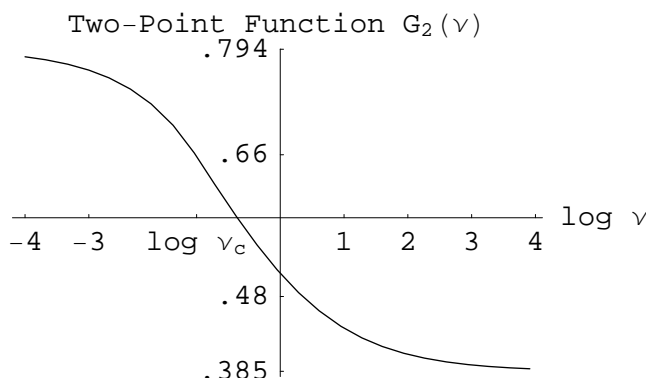


Figure 1: Two-point glue-ring expectation value $G_2(\nu)$ plotted against $\log \nu$. The asymptotic behavior for heavy quarks or weak-coupling ($\nu \rightarrow \infty$, 1-cut phase) is separated from the behavior for gluons strongly coupled to light quarks ($\nu \rightarrow 0$, 2-cut phase) by a third-order phase transition at ν_c . G_2 grows as we go from gluons in the vacuum ($\nu = \infty$) to those in a baryon strongly coupled to light quarks (small ν) since $\rho(x)$ transforms from uni-modal to bi-modal.

5.2 Free energy

Let us define the entropy [14] as

$$\chi = \mathcal{P} \int \int \rho(x)\rho(y) \log |x - y| dx dy. \tag{5.17}$$

The large- N free energy is the Legendre transform of entropy, $E = \int dx \rho(x) S(x) - \chi$. An expression for E involving only single integrals can be obtained using the Mehta-Dyson equation

$$\frac{1}{2} S'(z) = \mathcal{P} \int \frac{\rho(y)}{z - y} dy; \quad z \in \text{supp}(\rho). \tag{5.18}$$

Integrating with respect to z from x_0 to x , both of which lie in the support of ρ ,

$$\frac{1}{2} \left[S(x) - S(x_0) \right] = \mathcal{P} \int dy \rho(y) \left[\log |x - y| - \log |x_0 - y| \right]. \tag{5.19}$$

Now multiply by $\rho(x)$ and integrate with respect to x ,

$$\frac{1}{2} \int dx \rho(x) \left[S(x) - S(x_0) \right] = \mathcal{P} \int \int dx dy \rho(x)\rho(y) \left[\log |x - y| - \log |x_0 - y| \right]. \tag{5.20}$$

Thus

$$E = \frac{1}{2} S(x_0) + \mathcal{P} \int dx \rho(x) \left[\frac{1}{2} S(x) - \log |x - x_0| \right]$$

$$E(\nu) = \frac{1}{2} S(x_0) + \frac{1}{2} G_4 - \mathcal{P} \int_0^{2a} dx \rho(x) \log |(\nu + x^2)(x^2 - x_0^2)|. \tag{5.21}$$

x_0 is arbitrary provided it lies in the support of ρ . The x_0 independence of E follows from the Mehta-Dyson equation and was also verified numerically. $x_0 = 0$ is most convenient

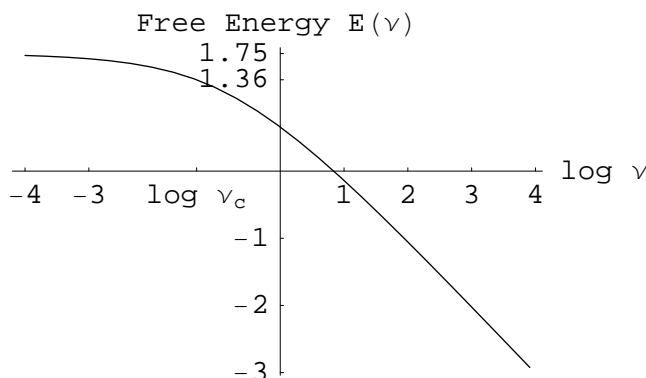


Figure 2: Free energy $E(\nu)$ versus $\log \nu$. The 2-cut phase lies to the left while the 1-cut phase lies to the right of the critical point at ν_c . For heavy quarks or weak-coupling ($\nu \rightarrow \infty$), $E(\nu) \rightarrow -\log \nu - .996$ as shown in section 7.1 and illustrated by the linear asymptotic behavior to the right in the plot. In the chiral limit (section 7.3), free energy approaches a constant $E(0) \approx 1.751$.

for us, so

$$E(\nu) = -\frac{1}{2} \log \nu + \frac{1}{2} G_4(\nu) - \mathcal{P} \int_0^{2a} \rho(x) \log(\nu x^2 + x^4) dx. \quad (5.22)$$

The new ingredient in free energy not determined by polynomial moments G_n is a sort of logarithmic moment. We could not evaluate it in terms of known functions for the 1-cut $\rho(x)$ (5.14), but the integration is easily performed numerically and plotted in figure 2.

5.3 Domain of validity of one-cut solution

For which $\nu \geq 0$ is the 1-cut solution valid? So far we have not imposed the $\rho \geq 0$ condition. The 1-cut solution

$$\rho(x) = -(1/\pi) \frac{P(x)}{(\nu + x^2)} \sqrt{4a^2 - x^2}; \quad P(x) = \alpha + \gamma x^2 + \epsilon x^4 \quad (5.23)$$

will break down if $\rho < 0$. The boundary of the region of validity is given by ν for which $\rho(x) = 0$ for some $|x| \leq 2a$. The most obvious way in which the 1-cut solution breaks down is when $\rho(0) = 0$ and we transit to a 2-cut solution without any support in an interval containing $x = 0$. $\rho(0) = -\frac{2a\alpha}{\pi\nu} = 0$ implies $\alpha = 0$, since $a > 0$. $\alpha = 0 \Rightarrow 6a^4 + 2\nu a^2 - 1 = 0$. Regarded as a quadratic in a^2 , the unique positive solution is

$$a_c^2 = -\frac{\nu}{6} + \frac{\sqrt{\nu^2 + 6}}{6} \geq 0 \quad \text{for } \nu > 0. \quad (5.24)$$

This is the critical value of $a(\nu)$ at which the 1-cut solution vanishes at the origin. Substituting a_c^2 into the quintic (5.13) we find a condition on the transition point $\nu = \nu_c$

$$\nu \left[-27 - 8\nu^4 + 48\nu \sqrt{(6 + \nu^2)} + 8\nu^2(-9 + \nu \sqrt{(6 + \nu^2)}) \right] = 0. \quad (5.25)$$

Assuming $\nu > 0$, the only possibility is for the factor in parentheses to vanish. Upon simplification it becomes a quadratic equation $48\nu^4 + 368\nu^2 - 27 = 0$. The positive solution is

$$\nu_c = \sqrt{\frac{13\sqrt{13} - 46}{12}} \approx 0.27. \tag{5.26}$$

Are there any other transition points, i.e. does $\rho(x)$ become negative for any $0 < |x| \leq 2a$? For example we find that there is no value of $\nu > 0$ for which $P(2a) = 0$. Indeed, the condition $P(2a) = 0$ implies that $30a^4 + 6\nu a^2 - 1 = 0$ or

$$a^2 = -\frac{\nu}{10} + \frac{\sqrt{9\nu^2 + 30}}{30}. \tag{5.27}$$

However, when this is substituted into the quintic (5.13), the condition

$$4875\nu - 600\nu^3 + 72\nu^5 = \left(24\nu^4 - \frac{3200}{3} - 240\nu^2\right) \sqrt{30 + 9\nu^2} \tag{5.28}$$

has no solution for $\nu > 0$. Based on the shape of $S(x)$ we expect the 1-cut solution to be valid for all $\nu > \nu_c$, and we have checked that this is indeed the case.

6. Two-cut solution of quartic + log matrix model

For small ν , $S(x) = x^4 - \log[\nu + x^2]$ develops a repulsive core near $x = 0$. For $\nu < \nu_c$, we expect the 1-cut solution of the Mehta-Dyson equation

$$S'(x) = 4x^3 - \frac{2x}{\nu + x^2} = 2\mathcal{P} \int \frac{\rho(y)}{x - y} dy, \quad x \in \text{supp}(\rho) \tag{6.1}$$

to make a transition to a 2-cut solution supported on $[-2a, -2b] \cup [2b, 2a]$. The generating function of moments $F(z) = \int \frac{\rho(y)}{z - y} dy$ enjoys the same properties (a)-(e) as before with $[-2a, 2a]$ replaced with $[-2a, -2b] \cup [2b, 2a]$ where $a > b \geq 0$. An appropriate ansatz for $F(z)$ is

$$\begin{aligned} F(z) &= \frac{1}{2}S'(z) + R(z)\sqrt{(z^2 - 4a^2)}\sqrt{(z^2 - 4b^2)} \\ &= 2z^3 - \frac{z}{(\nu + z^2)} + \frac{(\beta z + \delta z^3)}{(\nu + z^2)}\sqrt{(z^2 - 4a^2)}\sqrt{(z^2 - 4b^2)} \end{aligned} \tag{6.2}$$

β, δ, a, b have to be fixed using analyticity and asymptotic behavior of $F(z)$. As $|z| \rightarrow \infty$,

$$\begin{aligned} F(z) \rightarrow & \left(2 + \delta\right)z^3 + \left(\beta - \delta(2a^2 + 2b^2 + \nu)\right)z \\ & + \left(-1 - 2\beta(a^2 + b^2) - 2\delta(a^2 - b^2)^2 - \nu(\beta - 2\delta(a^2 + b^2)) + \delta\nu^2\right)\frac{1}{z} + \mathcal{O}\left(\frac{1}{z^3}\right) \end{aligned} \tag{6.3}$$

The requirement $F(z) \sim \frac{1}{z} + \mathcal{O}(1/z^3)$ implies

$$\delta = -2; \quad \beta = -2(2a^2 + 2b^2 + \nu) \quad \text{and} \quad 6(a^4 + b^4) + 4a^2b^2 + 2\nu(a^2 + b^2) - 1 = 0. \tag{6.4}$$

The condition that $F(z)$ be analytic at $z = \pm i\sqrt{\nu}$ implies $(\beta - \nu\delta)\sqrt{(\nu + 4a^2)(\nu + 4b^2)} + 1 = 0$. Substituting for β and δ , we are left with a pair of algebraic equations for a and b .

$$\begin{aligned} 6(a^2 + b^2)^2 - 8a^2b^2 + 2\nu(a^2 + b^2) - 1 &= 0 \\ 4(a^2 + b^2)\sqrt{16a^2b^2 + 4\nu(a^2 + b^2) + \nu^2} - 1 &= 0. \end{aligned} \quad (6.5)$$

Let $s = a^2 + b^2$ and $p = a^2b^2$ be the sum and product, then

$$6s^2 - 8p + 2\nu s - 1 = 0 \quad \text{and} \quad 4s\sqrt{16p + 4\nu s + \nu^2} - 1 = 0. \quad (6.6)$$

We can eliminate

$$p = \frac{1}{4} \left[3s^2 + \nu s - \frac{1}{2} \right] \quad (6.7)$$

and get an algebraic equation for s , $4s\sqrt{12s^2 + 8\nu s + (\nu^2 - 2)} = 1$, which has at most one positive solution s for $\nu > 0$. Squaring it we get a quartic equation

$$12s^4 + 8\nu s^3 + (\nu^2 - 2)s^2 - \frac{1}{16} = 0. \quad (6.8)$$

This can be solved in lengthy but closed form and the unique positive s selected. From s , we get p as well and

$$2a = \sqrt{2s + 2\sqrt{s^2 - 4p}}, \quad 2b = \sqrt{2s - 2\sqrt{s^2 - 4p}}. \quad (6.9)$$

The eigenvalue density, supported on $2b \leq |x| \leq 2a$ is

$$\begin{aligned} \rho(x) &= -\frac{R(x)}{\pi} \sqrt{(4a^2 - x^2)(x^2 - 4b^2)} \\ &= \frac{(2x^3 + (4a^2 + 4b^2 + 2\nu)x)}{\pi(\nu + x^2)} \sqrt{(4a^2 - x^2)(x^2 - 4b^2)}. \end{aligned} \quad (6.10)$$

Using the Laurent expansion of $F(z)$, we get the moments $G_{2n+1} = 0$,

$$\begin{aligned} G_2 &= 16s^3 - 64ps - 8\nu s^2 - 4\nu^2 s + \nu, \\ G_4 &= 64p^2 + 36s^4 - 32ps(5s - \nu) - 8s^3\nu - \nu^2 + 8s^2\nu^2 + 4s\nu^3, \quad \dots \end{aligned} \quad (6.11)$$

s and p can be eliminated using the solution of the quartic equation (6.8). The results are plotted in figure 1 and 3.

6.1 Free Energy

The free energy may be expressed in terms of single integrals

$$E(\nu) = \frac{1}{2}S(x_0) + \frac{1}{2}G_4 - \frac{1}{2}\mathcal{P} \int dx \rho(x) \log |(\nu + x^2)(x^2 - x_0^2)| \quad (6.12)$$

where x_0 is any point in the support of ρ . To obtain (6.12) we need to pay attention to the fact that the Mehta-Dyson equation (6.1) is valid in two disjoint intervals. Begin by

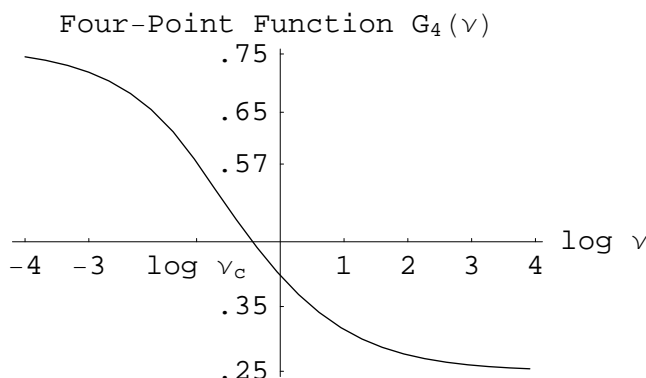


Figure 3: Four-point glue-ring expectation value $G_4(\nu)$ versus $\log \nu$. The 1-cut solution for $\nu \geq \nu_c$ and 2-cut solution for $\nu \leq \nu_c$ are not analytic continuations of each other despite appearances. The asymptotic values at $\nu = 0, \infty$ and at ν_c shown on the vertical axis are obtained analytically in section 7.

integrating the Mehta-Dyson equation with respect to z from x_+ to x where $x_+, x \in [2b, 2a]$ to get⁵

$$\frac{1}{2}[S(x) - S(x_+)] = \int dy \rho(y) [\log |x - y| - \log |x_+ - y|] \quad \text{for } x, x_+ \in [2b, 2a] \quad (6.13)$$

Multiplying by $\rho(x)$ and integrating with respect to x from $2b$ to $2a$ and simplifying gives

$$\int_{2b}^{2a} dx \int dy \rho(x) \rho(y) \log |x - y| = \frac{1}{2} \int_{2b}^{2a} dx \rho(x) S(x) - \frac{S(x_+)}{4} + \frac{1}{2} \int dy \rho(y) \log |x_+ - y| \quad (6.14)$$

for $x_+ \in [2b, 2a]$. When the limits of integration are not specified, the integral is over $\text{supp}(\rho)$. Similarly, for $x_- \in [-2a, -2b]$ we get

$$\int_{-2a}^{-2b} dx \int dy \rho(x) \rho(y) \log |x - y| = \frac{1}{2} \int_{-2a}^{-2b} dx \rho(x) S(x) - \frac{S(x_-)}{4} + \frac{1}{2} \int dy \rho(y) \log |x_- - y|. \quad (6.15)$$

Adding these two, we get for the entropy (5.17),

$$\chi = \frac{1}{2} \int dx \rho(x) S(x) - \frac{S(x_+)}{4} - \frac{S(x_-)}{4} + \frac{1}{2} \int dy \rho(y) \log |(x_+ - y)(x_- - y)| \quad (6.16)$$

where x_{\pm} are in the positive and negative part of $\text{supp}(\rho)$. By choosing $x_- = -x_+$ we simplify matters using the fact that $S(x)$ is an even function

$$\chi = \frac{1}{2} \int \rho(x) S(x) dx - \frac{1}{2} S(x_+) + \frac{1}{2} \int dx \rho(x) \log |x^2 - x_+^2|. \quad (6.17)$$

⁵The principle value prescription is implied where necessary and for brevity will not be explicitly indicated.

Now observe that in this expression, the sign of x_+ does not matter, so we can call $x_0 = x_+$ and pick it anywhere in $\text{supp}(\rho)$. Recalling that $E = \int \rho(x)S(x)dx - \chi$ we get the advertised expression (6.12) for the 2-cut free energy. Though the complexity in evaluating $E(\nu)$ has been reduced, we have not been able to find the above logarithmic moment in terms of known functions. The numerically evaluated free energy $E(\nu)$ is plotted in figure 2.

6.2 Phase transition to one-cut solution

We expect the 2-cut solution to make a transition to the 1-cut solution when the intervals $[-2a, -2b], [2b, 2a]$ merge, i.e. $b = 0$, which implies $p = 0$ and $s = a^2$. Inserting in (6.7) gives a quadratic equation $3s_c^2 + \nu s_c - \frac{1}{2} = 0$ whose positive solution $s_c = a_c^2 = -\frac{\nu}{6} + \frac{\sqrt{\nu^2+6}}{6}$ is the same as the value of a^2 at which the 1-cut solution breaks down. When s_c is substituted in (6.8), we get the same phase transition point $\nu = \nu_c$ as before (5.26). The 2-cut solution for $\nu < \nu_c$ takes over when the 1-cut solution breaks down.

7. Special cases: weak-coupling, critical point and chiral limit

7.1 Heavy quark or weak-coupling limit $\nu \rightarrow \infty$

In our toy-model, when the quarks are very heavy or the coupling constant is small, the self-interactions of the gluons dominates their interactions with the quarks in the baryon. This is the limit $\nu \rightarrow \infty$ where the action $S(A) \rightarrow \text{tr} A^4$ up to an additive constant. It is as if the gluons do not feel the presence of the baryon and we return to calculating *vacuum* correlations of glue-ring observables. This limit lies in the deep end of the 1-cut phase, where calculations simplify. The quintic equation (5.13) for the limits of $\text{supp}(\rho)$ reduces to a quadratic equation $48s^4 - 16s^2 + 1 = 0$ whose physical solution is $s = \sqrt{3}/6$. The limiting eigenvalue density is

$$\rho(x, \nu \rightarrow \infty) = \frac{1}{\pi}(4s + 2x^2)\sqrt{4s - x^2}, \quad |x| \leq 2\sqrt{s}, \quad (7.1)$$

The odd moments vanish while the even moments and free energy (5.21) are

$$G_{2n} = \frac{2^{n+1}(n+1)\Gamma(n+\frac{1}{2})}{3^{n/2}\sqrt{\pi}\Gamma(n+3)}; \quad G_2 = \frac{2\sqrt{3}}{9} \approx .385, \quad G_4 = \frac{1}{4}, \dots \quad (7.2)$$

$$E = -\log \nu + \frac{1}{2}G_4 - \frac{1}{2} \int dx \rho(x) \log x^2 = -\log \nu + \frac{1}{8}(3 + \log 144) \approx -\log \nu + .996.$$

These limiting values are seen to agree with the numerically obtained behavior of $E(\nu)$, $G_2(\nu)$ and $G_4(\nu)$ plotted in the figure 2, 1 and 3 for a wide range of values of ν .

7.2 Neighborhood of the phase transition

At ν_c , the 1-cut solution, valid for high quark mass m or weak-coupling α makes a phase transition to a 2-cut solution, which is valid for low quark mass or strong-coupling ($\nu \leq \nu_c$). In the immediate vicinity of the phase transition, observables are more easily evaluated

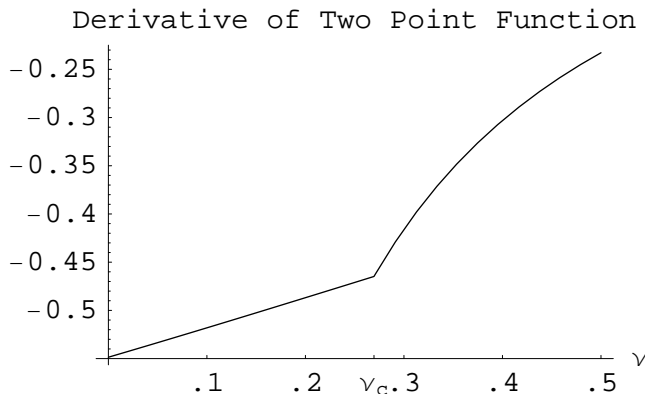


Figure 4: Derivative of the two-point correlation $G_2(\nu)$ has a kink at $\nu_c = 0.27$ indicating that its second derivative is discontinuous.

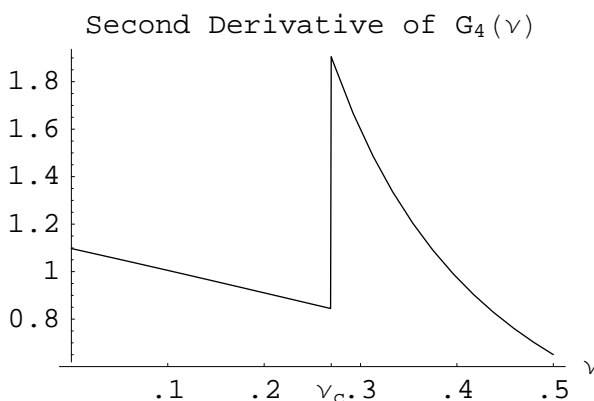


Figure 5: Jump discontinuity in the second derivative of $G_4(\nu)$ at the critical point ν_c .

than generically. We find that the eigenvalue density is continuous across the transition, the critical eigenvalue density is (see figure 6)

$$\rho_c(x) = \frac{\{(7\sqrt{13} - 22)\nu_c x^2 + (13\sqrt{13} - 46)x^4\}}{18\pi\nu_c^2(\nu_c + x^2)} \sqrt{(8\sqrt{13} + 20)\nu_c - 9x^2} \quad (7.3)$$

and is supported on $[-2a_c, 2a_c]$ where $a_c^2 = \sqrt{(\frac{\sqrt{13}-2}{12})}$, $a_c \approx .605$. Thus, all moments and the free energy are also continuous. The next question is whether their derivatives are discontinuous across the transition. We find that the second derivatives of G_2 and G_4 have jump discontinuities across the phase transition. We have calculated this behavior analytically. To find the derivatives of the correlations (5.16), (6.11) at ν_c we need the values and derivatives of a, s and p at ν_c . The critical values $\nu_c = \frac{\sqrt{13\sqrt{13}-46}}{12}$, $s_c = a_c^2$, $p_c = 0$ have already been determined without much trouble. For their derivatives, it is again not necessary to solve the quartic (6.8) and quintic (5.13) equations, but suffices to solve linear equations obtained by differentiating these at ν_c . For example, suppose we want $G_2'(\nu_c^-)$, which corresponds to the approach from the 2-cut phase. From (6.11)

$$G_2'(\nu) = 48s^2s' - 64ps' - 64sp' - 4\nu^2s' - 8s\nu - 8s^2 - 16ss'\nu + 1 \quad (7.4)$$

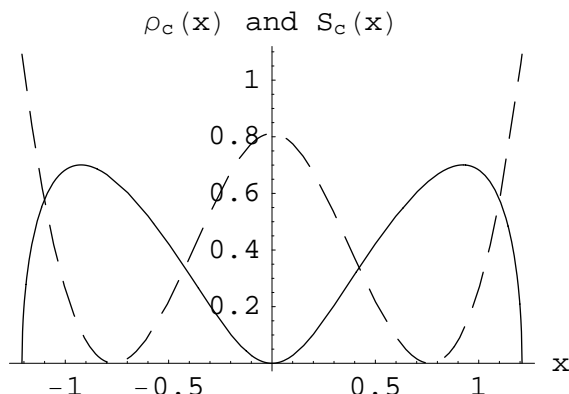


Figure 6: Eigenvalue density at the critical point $\rho_c(x)$, the continuous curve. The dashed curve is the critical action $S(x, \nu_c)$ plus a constant chosen to make it vanish at its minima. While ρ is peaked around the minima of S , the spread allows it to maximize entropy χ (5.17).

Differentiating (6.8) and solving the linear equation for $s'(\nu)$ gives

$$s'(\nu) = \frac{-\nu s - 4s^2}{-2 + \nu^2 + 12\nu s + 24s^2}. \quad (7.5)$$

$p'(\nu)$ is determined similarly using (6.7) and evaluated at ν_c . In this manner we get

$$\begin{aligned} G_2(\nu_c^\pm) &= \frac{(13\sqrt{13} + 19)}{27} \nu_c \approx .66; & G_2'(\nu_c^\pm) &= \frac{\sqrt{13} - 5}{3} \approx -.46 \\ G_2''(\nu_c^+) &= \frac{4}{9} \sqrt{\frac{(205\sqrt{13} - 122)}{39}} \approx 1.77; & G_2''(\nu_c^-) &= \frac{8}{51} \sqrt{\frac{(1669\sqrt{13} - 5858)}{39}} \approx 0.32 \end{aligned} \quad (7.6)$$

and

$$\begin{aligned} G_4(\nu_c) &= \frac{(26\sqrt{13} + 119)}{27} \nu_c^2 \approx .57; & G_4'(\nu_c) &\approx .53 \\ G_4''(\nu_c^-) &= \frac{(4706 - 1150\sqrt{13})}{663} \approx 0.844; & G_4''(\nu_c^+) &= \frac{(2522 - 514\sqrt{13})}{351} \approx 1.905 \end{aligned} \quad (7.7)$$

These agree with the numerically determined correlations plotted in figure 4 and figure 5.

The free energy $E(\nu)$ and its first two derivatives are continuous at ν_c . $E'''(\nu)$ has a jump discontinuity at ν_c . For example, to calculate $E'(\nu_c^+)$, we differentiate the integral representation for 1-cut free energy (5.21)

$$\begin{aligned} E(\nu) &= -\frac{1}{2} \log \nu + \frac{1}{2} G_4 - \int_0^{2a} dx \rho_\nu(x) \log(\nu x^2 + x^4) \\ \Rightarrow E'(\nu) &= -\frac{1}{2\nu} + \frac{1}{2} G_4'(\nu) - \int_0^{2a} dx \left\{ \frac{\rho(x)}{\nu + x^2} + \frac{\partial \rho}{\partial \nu} \log(\nu x^2 + x^4) \right\}. \end{aligned} \quad (7.8)$$

$\frac{\partial \rho}{\partial \nu}$ can be got from the explicit formula (5.14). We omitted the term involving the derivative of the upper limit of integration because $\rho(2a) = 0$. Evaluating at ν_c , using the above result for $G_4'(\nu)$ and doing the integral numerically gives us $E'(\nu_c^+)$. Proceeding along these lines we get $E(\nu_c) \approx 1.36$, $E'(\nu_c) \approx -1.32$, $E''(\nu_c) \approx .93$, $E'''(\nu_c^-) \approx -.7$, $E'''(\nu_c^+) \approx 7.05$. This is illustrated in figure 7. We conclude that the phase transition is of third-order.

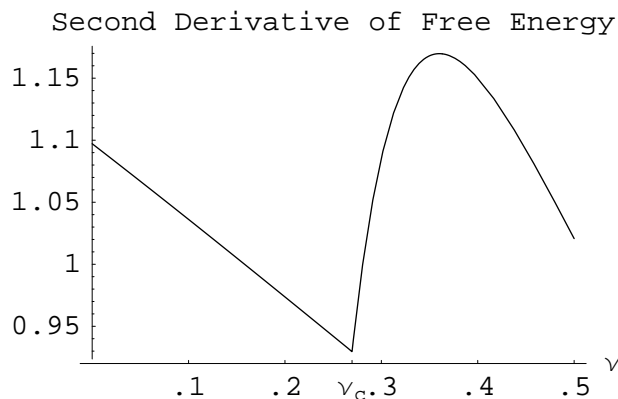


Figure 7: $E''(\nu)$ versus ν . The 3rd derivative of free energy is discontinuous at $\nu_c = 0.27$.

7.3 Chiral limit or strong-coupling limit $\nu \rightarrow 0$

In the limit of massless quarks or strong-coupling, $\nu \rightarrow 0^+$, which lies in the 2-cut phase. The action becomes $S(A) = \text{tr}[A^4 - \log(A^2)]$ and the gluons maximally feel the presence of quarks in the baryon. The solution again simplifies. The quartic equation for s (6.8) becomes a quadratic with solution $s = \frac{1}{2}\sqrt{\frac{2+\sqrt{7}}{6}}$ and $p = \frac{\sqrt{7}-2}{32}$. From this we get

$$\rho(x) = \frac{(2x^2 + 4s)}{\pi x} \sqrt{4sx^2 - x^4 - 16p} \quad (7.9)$$

and moments $G_2 = \frac{(4-\sqrt{7})}{3} \sqrt{\frac{4+2\sqrt{7}}{3}} \approx .794$ and $G_4 = \frac{3}{4}$ which are shown in figure 1 and 3. The free energy (6.12) has a finite limit $E(\nu = 0) \approx 1.751$ as illustrated in figure 2.

8. Discussion

We have studied a very simple matrix model for glue-ring correlations in a baryon in the limit of many colors. It was obtained as a caricature of the dimensional reduction of QCD to 1 + 1 dimensions. Our main finding is that there is a third-order phase transition that separates a phase where gluons are weakly coupled to heavy quarks ($\nu \geq \nu_c$) from one where the quarks are light and strongly coupled to gluons ($0 \leq \nu \leq \nu_c$). ν , a dimensionless ratio of quark mass to coupling constant is the only parameter of the model. While for $\nu \geq \nu_c$ we have a 1-cut solution of the matrix model, for $\nu \leq \nu_c$ we have a 2-cut solution. The gauge-invariant observables (glue-ring expectation values) are described in these two phases by two different analytic functions of ν that disagree beyond their first derivatives at ν_c (See figure 1, 3, 4, 5). Moreover, the case of gluons in the vacuum i.e. where the baryon is absent, corresponds to $\nu \rightarrow \infty$, which is deep inside the 1-cut phase. The physically interesting value of ν most likely is small and lies in the 2-cut phase, since current quarks in the proton are very light compared to Λ_{QCD} . Thus, the vacuum correlations of gluons are likely to be separated by a phase transition from those in a baryon state. Moreover, from figure 1 and 3 we see that gluon correlations are enhanced inside the baryon compared to their values in the vacuum ($\nu = \infty$). This reflects the growth of moments of the distribution

of eigenvalues, as they make a transition from being clustered about the origin to being supported on a pair of intervals excluding the origin. Though this is very far from explaining why about half the proton's momentum is contributed by gluons, it does indicate that the qualitative features of gluon correlations in the vacuum can be quite different from those in a baryon state. Even if we could use a weak-coupling expansion to describe a bound-state like a baryon, the phase transition would invalidate analytic continuation to the physically relevant baryon containing gluons strongly coupled to light quarks.

The wider applicability of our results is called into question by our approximations and truncations. The sharp phase transition is an artifact of $N = \infty$. For finite N , the matrix model has finitely many degrees of freedom and cannot display non-analytic behavior. Nevertheless, this is probably the most benign of our approximations. The finite N theory should display qualitative differences between the two regimes. Absence of space-time derivatives and non-local interactions due to longitudinal gluons are the more significant shortcomings of our model. This toy-model has given us a cartoon of how the theory may behave as ν is varied. A hamiltonian approach may have a better chance at shedding light on the matrix field theory or matrix quantum mechanics version of this problem. We hope the proposal of treating gluons in a 'fixed baryon background' $|\Psi\rangle$ (2.4), along with other new ideas will help simplify the matrix field theory in order to better understand the emergent bound-state structure of gluons in a nucleon.

Some questions for future work are collected here. (i) Can we use a variational or other approximation method to understand this phase transition? Such an approach has a better chance of generalizing to multi-matrix models. (ii) Besides glue-ring correlations, we are also interested in open string correlations in a baryon state. Can we get a zero-dimensional toy-model for these as well? (iii) Can we shed any light on the multi-matrix model that arises when we do *not* assume the transverse gluon field to be equal at the positions of the quarks? (iv) What is the relation between the gluon distribution function extracted from experimental data and the gauge-invariant glue-ring variables that would come from solving the matrix field theory? One suspects that the gluon distribution is essentially determined by a two-point glue-ring expectation value. What is the simplest truncated form of the matrix field theory where such a gluon distribution function can be estimated? (v) In [14], we obtained algebraic and probabilistic characterizations of the entropy χ whose Legendre transform is the free energy of matrix models. However, the formula for χ in generic multi-matrix models is quite complicated. Is there any multi-matrix generalization of the trick (5.21) of using the Mehta-Dyson 'equation of motion' to reduce double integrals to single integrals? (vi) Can we find a variational principle that determines the closed and open string observables of our matrix field theory or some finite dimensional truncation thereof? Such a variational principle for closed string observables in multi-matrix models was found in [14].

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